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On the Inequalities of Integro-Fractional Differential Equations

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ABSTRACT: Inequalities of the integro–differential equations with non-integer order:

$$u^{(\alpha)}(t) = g(t,u(t)) + \int_{t_0}^{t} H(t,s(t),u(s))ds, \qquad \alpha \in \Re, \ \ 0 < \alpha \le 1$$

with the initial condition

$$u^{(\alpha-1)}(t_0) = u_0,$$

have been investigated. Our results are a generalization of previous known results.

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Lyapunov Stability Solutions of Integro-Fractional Differential Equations

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Abstract

Lyapunov stability and asymptotic stability conditions for the solutions of the integro–fractional differential equations:

$$x^{(\alpha)}(t) = f(t, x(t)) + \int_{t_0}^{t} K(t, s(t), x(s))ds, \quad \alpha \in \Re, \ 0 < \alpha \le 1$$

with the initial condition

$$x^{(\alpha-1)}(t_0) = x_0,$$

have been investigated. Our methods are applications of Gronwall's Lemma and Schwartz inequality.

1. INTRODUCTION

Consider the integro-differential equations with non-integer order of the type:

$$u^{(\alpha)}(t) = g(t, u(t)) + \int_{t_0}^{t} H(t, s(t), u(s))ds, \quad \alpha \in \Re, \ 0 < \alpha \le 1 \qquad (1.1)$$

with the initial condition:

$$u^{(\alpha-1)}(t_0) = u_0, \qquad (1.2)$$

where $\Re$ is the set of real numbers, $J = [t_0, t_0 + a]$, $a > 0$, $g \in C[J \times \Re^n, \Re^n]$, and $H \in C[J \times J \times \Re^n, \Re^n]$, where $\Re^n$ denotes the real n-dimensional Euclidean space, and u_0 is a real positive constant.

The integro-differential inequalities occupy a very privileged position in the theory of integro-differential equations. These inequalities have been greatly enriched by the recognition of their potential and intrinsic worth in many applications of the applied sciences. The theory of such inequalities depends essentially upon the integration of integro-differential inequalities which are usually known as the general comparison principles, see [6] and [11].

In recent years, there has been an interest in the study of integro-fractional differential equations. In [7], we used Schauder's fixed-point theorem to obtain local existence, and Tychonov's fixed-point theorem to obtain global existence of solution of (1.1) and (1.2). In [8], we used the successive approximations method and Arzela-Ascoli lemma to obtain existence and uniqueness of solution of equation (1.1) and (1.2). The existence of extremal (maximal and minimal) solutions of the integro-fractional differential equations (1.1) and (1.2) using comparison principle and Ascoli lemma have been investigated in [9].

On the other hands equations (1.1) and (1.2) are a generalization of the fractional differential equations:

$$u^{(\alpha)}(t) = g(t, u(t)), \quad \alpha \in \Re, \ 0 < \alpha \le 1, \qquad (1.3)$$

with the initial condition:

$$u^{(\alpha-1)}(t_0) = u_0 . \qquad (1.4)$$

The existence and uniqueness of the solution of (1.3) and (1.4), in additional to some analytical properties and important inequalities, are investigated in [4] and [5].

In this paper, we shell introduce a new type of inequalities called "integro-fractional differential inequalities". These inequalities may be considered as a generalization of the integro-differential inequalities in [6] and a generalization of the inequalities hold for (1.3) and (1.4). We shall obtain integro-differential inequalities results of equations (1.1) and (1.2), and a result concerning the upper and lower functions of solution of (1.1) and (1.2).

We shall adopt the definitions and notations used in [1] and [2].

Definition (1.1): Let f be a function which is defined almost everywhere (a.e.) on $[a,b]$. For $\alpha > 0$, we define

$$\overset{b}{\underset{a}{I}}{}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} f(s) ds , \qquad (1.5)$$

provided that this integral (Lebesgue) exists, where Γ is the Gamma function.

We also need the following result for completeness.

Lemma (1.1): The initial value problem (1.1) and (1.2) is equivalent to the nonlinear integral equation

$$u(t) = \frac{u_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} g(s,u(s)) ds + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \int_{s}^{t} H(\sigma,s,u(s)) d\sigma \, ds , \qquad (1.6)$$

where $0 < t_0 < t \leq t_0 + a$. In other words, every solution of the integral (1.6) is also a solution of our original initial value problem (1.1) and (1.2), and vice versa.

Proof: It can be proved easily by applying the integral operator (1.5) to both sides of (1.1), as we did in [4] , and using some classical results from fractional calculus in [1] to get (1.6).

2. THE MAIN THEOREMS

In this section, we shall prove our main results, we begin by proving the basic integro-fractional differential inequality of (1.1) satisfying (1.2).

Theorem (2.1): Assume that

(i) $g \in C[\Re_+ \times \Re, \Re]$, $H \in C[\Re_+^2 \times \Re, \Re]$ and $H(t,s,u)$ is monotone non-decreasing in u for each fixed $(t,s) \in \Re_+^2$ **;**

(ii) $v^{(\alpha)}(t) \le g(t,v(t)) + \int_{t_0}^{t} H(t,s(t),v(s))ds$ and

$$w^{(\alpha)}(t) \ge g(t,w(t)) + \int_{t_0}^{t} H(t,s(t),w(s))ds \;;$$

(iii) for $(t,s) \in \Re_+^2$, $x \ge y$ and $L \ge 0$,

$$g(t,x) - g(t,y) \le L(x-y), \qquad H(t,s,x) - H(t,s,y) \le L^2(x-y).$$

Then we have

$$v(t) \le w(t) \quad for \quad t \ge t_0, \quad provided \quad v^{(\alpha-1)}(t_0) \le w^{(\alpha-1)}(t_0). \tag{2.1}$$

Proof:

We shall first consider the result for strict inequalities. Assume that the conclusion of the theorem is false; then there exists a t_1 such that

$$v(t_1) = w(t_1), \quad v(t) < w(t), \quad t_0 \le t < t_1. \tag{2.2}$$

Clearly $t_1 > t_0$, because in this case $v^{(\alpha-1)}(t_0) < w^{(\alpha-1)}(t_0)$. Since H is monotone non-decreasing in u, it follows from (2.2) that $H(t_1,s,v(s)) \le H(t_1 s, w(s))$, $t_0 \le s \le t_1$. Also from (2.2), we get $v^{(\alpha)}(t_1) \ge w^{(\alpha)}(t_1)$. Hence, if one of the inequalities in (ii) is strict, we have:

$$w^{(\alpha)}(t_1) \le v^{(\alpha)}(t_1) = g(t,v(t_1)) + \int_{t_0}^{t_1} H(t_1,s,v(s))ds$$

$$\le g(t,w(t_1)) + \int_{t_0}^{t_1} H(t_1,s,w(s))ds < w^{(\alpha)}(t_1),$$

which is a contradiction. Hence it follows that $v(t) < w(t), \quad t \ge t_0$ is valid.

To prove the claim of the theorem, we assume, for $\varepsilon > 0$, that

$$w_\varepsilon(t) = w(t) + \varepsilon(2l)^{(1-\alpha)} e^{2Lt}. \tag{2.3}$$

From (2.1) and (2.3), we have

$$v^{(\alpha-1)}(t_0) \le w^{(\alpha-1)}(t_0) < w_\varepsilon^{(\alpha-1)}(t_0).$$

Also, by differentiating (2.3) with respect to α, see [1], and by using:

$$D^{\alpha}(e^{2Lt}) = (2L)^{\alpha} e^{2Lt}, \tag{2.4}$$

we obtain

$$w_{\varepsilon}^{(\alpha)}(t) = w^{(\alpha)}(t) + 2L\varepsilon e^{2Lt} \geq g(t, w(t)) + \int_{t_0}^{t} H(t, s, w(s))ds + 2L\varepsilon e^{2Lt}. \tag{2.5}$$

From condition (iii), we have

$$\left.\begin{array}{l} g(t, w_e(t)) - g(t, w(t)) \leq L\left[w_e(t) - w(t)\right] \\ H(t, s, w_e(t)) - H(t, s, w(t)) \leq L^2\left[w_e(t) - w(t)\right] \end{array}\right\}. \tag{2.6}$$

Hence from (2.3), (2.5) and (2.6), we get

$$w_{\varepsilon}^{(\alpha)}(t) \geq g(t, w_{\varepsilon}(t)) + \int_{t_0}^{t} H(t, s, w_{\varepsilon}(s))ds - \left(L\varepsilon e^{2Lt} + \int_{t_0}^{t} L^2 \varepsilon e^{2Ls} ds \right) + 2L\varepsilon e^{2Lt}$$

$$> g(t, w_{\varepsilon}(t)) + \int_{t_0}^{t} H(t, s, w_{\varepsilon}(s))ds, \quad t \geq t_0.$$

Now by the first part of the proof, we get $v(t) < w_{\varepsilon}(t), \quad for \quad t \geq t_0$, and letting $\varepsilon \to 0$, we obtain the state results. Hence the theorem is proved.

The next result is concerned with implicit integro-fractional differential inequalities which are of interest in some cases.

Theorem (2.2): Assume that:

(i) $F \in C\left[\Re_+ \times \Re^3, \Re\right]$ and $F(t, x, y, z)$ is non-decreasing in x for each (t, y, z) and non-decreasing in z for each (t, x, y);

(ii) The **integral** operator Tu defined by

$$Tu(t) = \frac{u_0}{\Gamma(\alpha)}(t - t_0)^{\alpha - 1} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} g(s, u(s))ds + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} \int_{s}^{t} H(\sigma, s, u(s))d\sigma\, ds$$

maps $C[\Re_+, \Re]$ into $C[\Re_+, \Re]$ and for $u_1, u_2 \in C[\Re_+, \Re]$, the inequalities

$u_1(t) \leq u_2(t), \quad t_0 \leq t \leq t_1, \quad t_0 \geq 0$, implies $Tu_1(t) \leq Tu_2(t), \quad for \quad t = t_1$;

(iii) $v, w \in C^1[\Re_+, \Re]$ and

$$F(t, v^{(\alpha)}, v, Tv) \le 0$$

$$F(t, w^{(\alpha)}, w, Tw) \le 0, \quad t \ge t_0,$$

one of the inequalities being strict.

Then $v^{(\alpha-1)}(t_0) < w^{(\alpha-1)}(t_0)$ *implies*

$$v(t) < w(t), \quad t \ge t_0. \tag{2.7}$$

Proof:
If the claim (2.7) is false, then there exists a $t_1 > t_0$ such that $v(t_1) = w(t_1)$ and $v(t) < w(t), \quad t_0 \le t < t_1$. This gives $v^{(\alpha)}(t_1) \ge w^{(\alpha)}(t_1)$.
It then follows from (ii) that $Tv \le Tw$ at $t = t_1$. Using the monotone character of F and (iii), we then get

$$0 \ge F(t_1, v^{(\alpha)}(t_1), v(t_1), Tv) \ge F(t_1, w^{(\alpha)}(t_1), w(t_1), Tw) \ge 0,$$

which is a contradiction.
Hence (2.7) is true and the proof is complete.

The following result shows that any solution of (1.1) and (1.2) can be bracketed between the lower and upper functions of this integro-fractional differential equations. These functions are defined, see [11], as follows:

Definition (3.1): The functions $v, w \in C^1[\Re_+, \Re]$ are said to be the lower and upper functions, respectively, with respect to the integro-fractional differential equations (1.1) and (1.2) if $v^{(\alpha)}(t)$ and $w^{(\alpha)}(t)$ exist and satisfy the inequalities:

$$v^{(\alpha)}(t) < g(t, v(t)) + \int_{t_0}^{t} H(t, s(t), v(s))ds,$$

$$w^{(\alpha)}(t) > g(t, w(t)) + \int_{t_0}^{t} H(t, s(t), w(s))ds,$$

on $0 < t_0 < t \le t_0 + a$.

Theorem (3.1): Assume that:

(i) $g \in C[\Re_+ \times \Re, \Re]$, $H \in C[\Re_+^2 \times \Re, \Re]$ and $H(t, s, u)$ is monotone non-decreasing in u for each fixed $(t, s) \in \Re_+^2$;

(ii) $v, w \in C^1[\Re_+, \Re]$ and be the lower and upper functions, respectively, with respect to (1.1) and (1.2) on $0 < t_0 < t \le t_0 + a$ and $u(t)$ be any solution of the equation (1.1) and (1.2) such that

$$v^{(\alpha-1)}(t_0) = u_0 = w^{(\alpha-1)}(t_0)\,. \tag{2.8}$$

Then, the inequality

$$v(t) < u(t) < w(t) \quad for \quad t > t_0\,. \tag{2.9}$$

holds.

Proof:

We shall first prove the right half of assertion (2.9). Set $m(t) = w(t) - u(t)$. Then, from the condition (2.8), it is clear that $m^{(\alpha)}(t_0) > 0$. This implies that $m(t)$ is increasing to the right of t_0 in a sufficiently small interval $[t_0, t_0 + \delta]$. Therefore, we have $u(t_0 + \delta) < w(t_0 + \delta)$. Since $w(t)$ is the upper function and u(t) is any solution of (1.1) and (1.2), it follows, from Definition (3.1), that for $t \in [t_0 + \delta, t_0 + a]$,

$$u^{(\alpha)}(t) = g(t, u(t)) + \int_{t_0}^{t} H(t, s(t), u(s))ds\,,$$

$$w^{(\alpha)}(t) > g(t, w(t)) + \int_{t_0}^{t} H(t, s(t), w(s))ds\,.$$

Thus, the application of Theorem (2.1) yields

$$u(t) < w(t) \quad for \quad t \in (t_0, t_0 + a).$$

The Proof for the left half of (2.9) is similar. Hence the proof is complete.

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