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Abstract. This paper presents a numerical method based on neural network, for solving the Lane-Emden equations singular initial value problems. The numerical solution is given for integer case and non integer case. The non integer case is taken in the sense of Riemann-Liouville operators.

Keywords: Neural network, fractional calculus, fractional differential equation, numerical solution.

INTRODUCTION

The theory of singular boundary value problems has become an important area of investigation in the past three decades (see [1-5]). One of the equations describing this type is the Lane-Emden equation. The Lane-Emden type equation was first published by Jonathan Homer Lane in 1870 [6] and further explored in detail by Emden [7] and has significant applications, is a second- order ordinary differential equation with an arbitrary index, known as the polytropic index, involved in one of its terms. The Lane-Emden equation describes a variety of phenomena in physics and astrophysics, including aspects of stellar structure, the thermal history of a spherical cloud of gas, isothermal gas spheres, and thermionic currents [8].

Solving the Lane-Emden problem, as well as other various linear and nonlinear singular initial value problems in quantum mechanics and astrophysics, is numerically challenging because of the singularity behavior at the origin. The approximate solutions to the Lane-Emden equation were given by homotopy perturbation method [9], variational iteration method [10], and Sinc-Collocation method [11], an implicit series solution [12]. Recently, Parand et. al [13] have proposed an approximation algorithm for the solution of the nonlinear Lane-Emden type equation using Hermite functions collocation method. Moreover, Adibi and Rismani [14] have introduced a modified Legendre-spectral method. Finally, Bhrawy and Alofi [15] have imposed a Jacobi-Gauss collocation method for solving nonlinear Lane-Emden type equations.

Lane-Emden equations have the following form

$$u''(t) + \frac{a}{t}u'(t) + f(t,u) = g(t),$$
$$0 < t \le 1, a \ge 0 \tag{1}$$

with the initial condition

$$u(0) = A, \quad u'(0) = B,$$

where A, B are constants, $f(t,u)$ is a continuous real valued function and $g(t) \in C[0,1]$.

FRACTIONAL CALCULUS

Fractional calculus and its applications (the theory of derivatives and integrals of any arbitrary real or complex order) has importance in several widely diverse areas of mathematical physical and engineering sciences. It generalizes the ideas of integer order differentiation and n-fold integration. Fractional derivatives introduce an excellent instrument for the description of general properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with classical integer-order models, in which such effects are in fact are neglected. The advantages of fractional derivatives become

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apparent in modeling mechanical and electrical properties of real materials, as well as in the description of properties of gases, liquids and rocks, and in many other fields [16].

The class of various types of fractional differential equations plays important roles, not only in mathematics but also in physics, control systems, dynamical systems and engineering to create the mathematical modeling of many physical phenomena. Naturally, such equations required to be solved. In the past three decades, many studies have been conducted on fractional calculus and fractional differential equations, involving different operators such as Riemann-Liouville operators [17], Erdelyi-Kober operators [18], Weyl-Riesz operators [19], Caputo operators [20] and Grunwald-Letnikov operators [21]. The existence of positive solution and multi-positive solutions for nonlinear fractional differential equation are established and studied [22]. Moreover, by using the concepts of the subordination and superordination of analytic functions, the existence of analytic solutions for fractional differential equations in complex domain are suggested and posed in [23,24].

One of the most frequently used tools in the theory of fractional calculus is furnished by the Riemann-Liouville operators (see[21]). The Riemann-Liouville fractional derivative could hardly pose the physical interpretation of the initial conditions required for the initial value problems involving fractional differential equations. Moreover, this operator possesses advantages of fast convergence, higher stability and higher accuracy to derive different types of numerical algorithms .

Definition 1. The fractional (arbitrary) order integral of the function f of order α 0 is defined by

$$I_a^\alpha f(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)d\tau.$$

When $a = 0$, we write $I_a^\alpha f(t) = f(t) * \phi_\alpha(t)$, where $(*)$ denoted the convolution product (see [22]), $\phi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, t > 0$ and $\phi_\alpha(t) = 0,\ t \le 0$ and $\phi_\alpha \to \delta(t)$ as $\alpha \to 0$ where $\delta(t)$ is the delta function.

Definition 2. The fractional (arbitrary) order derivative of the function f of order $0 \le \alpha < 1$ is defined by

$$D_a^\alpha f(t) = \frac{d}{dt}\int_a^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f(\tau)d\tau = \frac{d}{dt} I_a^{1-\alpha} f(t).$$

Remark 1. From Definition 1, and Definition 2, we have

$$D^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha},\ \ \mu > -1;\ \ 0 < \alpha < 1$$

and

$$I^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha},\ \ \mu > -1;\ \ \alpha > 0.$$

In this note, we consider the fractional Lane-Emden equations of the following form

$$D^\alpha u(t) + \frac{a}{t} u'(t) + f(t,u) = g(t),\quad 0 < t \le 1, a \ge 0,\ 1 < \alpha \le 2, \qquad (2)$$

with the initial condition

$$u(0) = 0,\quad u'(0) = 0,$$

where $f(t,u)$ is a continuous real valued function and $g(t) \in C[0,1]$.

ANALYTIC SOLUTION OF FRACTIONAL LAN-EMDEN EQUATION

Solution of ordinary type

In this section, we have illustrated the solution of Eq. (1), which can be found in [25] as follows: Consider the integral operator

$$L_a(.) = \int_0^t t^{-a} \int_0^t \tau^a (.) d\tau dt.$$

Applying L_{-a} on (1) we have

$$u(t) = A + Bt + L_a(g(t)) - L_a(f(t,u)).$$

Assume

$$G(t) = A + Bt + L_a(g(t))$$

$$F(t,u(t)) = t^{-a} \int_0^t (f(t,\mathrm{u}))dt.$$

Thus we have

$$u(t) = G(t) + \int_0^t F(t,u(t)),$$

which is nonlinear Volterra integral equation.

Solution of arbitrary type

In this section, we have illustrated the solution of Eq. (2). Assume that B is a Banach space of all continues bounded functions endow with the sup norm. By using some properties of the Riemann-Liouville fractional operators which are given in [21] , we have the following result:

Lemma 1. Let u be continues for all $t \in [0,1]$. Assume the problem

$$D^\alpha u(t) + \frac{a}{t} u'(t) = F(t,u),$$
$$0 < t \le 1, a \ge 0,\ 1 < \alpha \le 2, \qquad (3)$$

where

$F(t,u) = g(t) - f\big(t,u(t)\big)$ then

$$u(t) = h(t,u(t)) + I^{\alpha} F(t,u(t)),$$

where

$$h(t,u) := I^{\beta} \frac{u(t)}{t} \text{ and } 0 < \beta \leq 1.$$

Theorem 1. Let $F(t,u)$ defined in Lemma 1. If

$$\Gamma(\beta+1) > ca, \quad 0 < \beta \leq 1,$$

where $a >, c > 0$ then Eq.(3) has a solution.

Proof. Define an operator $P: \mathcal{B} \to \mathcal{B}$

$$(Pu)(t) = h(t,u(t)) + I^{\alpha} F(t,u(t))$$

where $h(t,u(t))$ defined in Lemma 1. Then we have

$$\begin{aligned} |(Pu)(t)| &= |h(t,u(t)) + I^{\alpha}\mathrm{F}(t,u(t))| \\ &\leq |h(t,u(t))| + |I^{\alpha} F(t,u(t))| \\ &\leq \frac{ac|u(t)|}{\Gamma(\beta+1)} + \frac{|F(t,u(t))|}{\Gamma(\alpha+1)} \end{aligned}$$

where

$$c := \frac{1}{t}, t \in (0,1].$$

Taking the sup, we obtain

$$\| P \| \leq \frac{\frac{\|F\|}{\Gamma(\alpha+1)}}{1 - \frac{ac}{\Gamma(\beta+1)}} := r.$$

Define the set $S := \{u \in \mathcal{B} : \| u \| \leq r, r \geq 0\}$, thus $P: S \to S$. The Arzela-Ascoli theorem ensures that every sequence of functions from $P(S)$ has got a uniformly convergent subsequence, and therefore $P(S)$ is relatively compact. *Schauder's* fixed point theorem asserts that P has a fixed point. By construction, a fixed point of P is a solution of the initial value problem (3).

NUMERICAL SOLUTION OF LANE-EMDEN EQUATION

In this section, we have illustrated some numerical solutions for two classes of Lane-Emden equations.

$$u''(t) + \frac{2}{t} u'(t) = 2(2\mathrm{t}^2 + 3)u(t), \quad 0 < t \leq 1 \qquad (4)$$

subject to the initial conditions

$$u(0) = 1, \quad u'(0) = 1.$$

The exact solution is $u(t) = e^{t^2}$ [25].

And

$$D^{\alpha} u(t) + \frac{2}{t} u'(t) = 2(2t^2 + 3)u(t), \quad 0 < t \leq 1 \qquad (5)$$

where $1 < \alpha \leq 2$. Eq.(5) has exact solution $E_{\alpha}(t^{\alpha})$, where E_{α}, is the Mittag-Leffler function, which is similar to the exponential function frequently used in the solutions of integer-order systems; it is defined as

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n\alpha+1)}$$

where $\alpha > 0$ and Γ is the Gamma function [22].

Artificial Neural Network

With an attempt to model, certain capabilities of the human's brain, Warren McCulloch and Walter Pitts have established a simplified model of a biological neuron in 1943, called the McCulloch-Pitts model, consisting of multiple inputs and one output. Neural networks have been successfully applied to a variety of real world classification tasks in industry, business, and sciences [26].

In this work, a standard back-propagation neural network (NN) is used to estimate the exact solution for the given fractional equation. The network consists of three layers; the first layer consists of neurons that are responsible for input data vectors into the neural network. The second layer is a hidden layer. This layer allows neural network to perform the error reduction, which is necessary to successfully achieve the desired output. The final layer is the output layer which is determined by the size of the set of desired outputs, which represent the estimated exact solution. Each possible output is represented by a separate neuron. There is one output from neural network. The neural network structure is shown in Fig. 1.

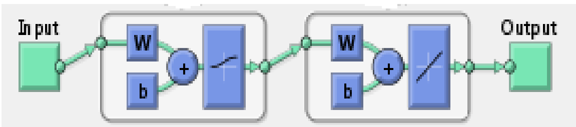

FIGURE 1. Neural Network structure

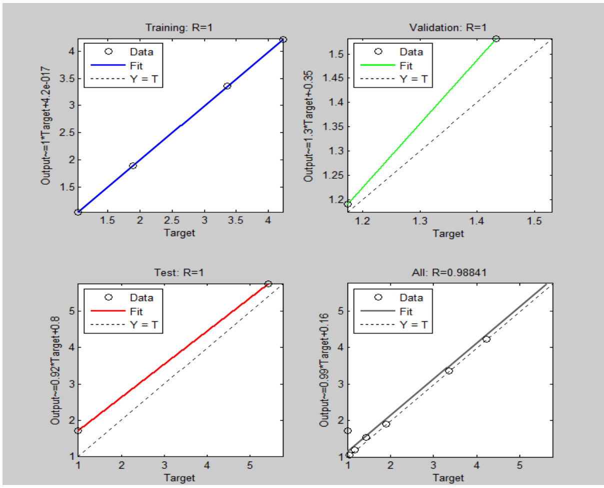

FIGURE 2. Regressions analysis (Ordinary case)

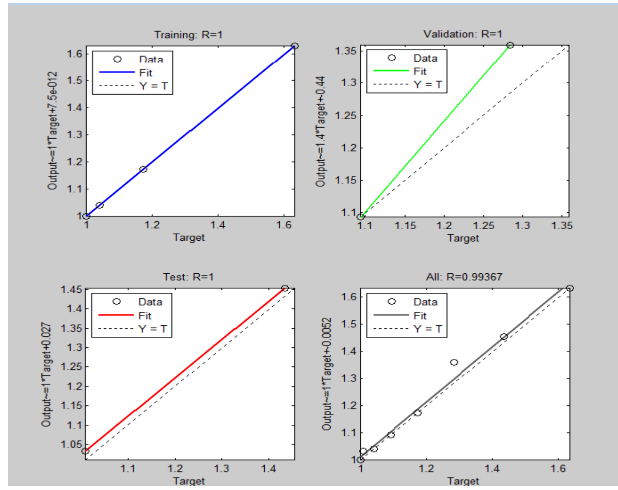

FIGURE 3. Regressions analysis(Fractional case)

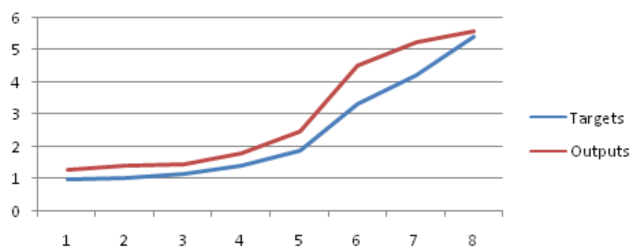

FIGURE 4. Tested outputs inputs with the desired targets (Ordinary case)

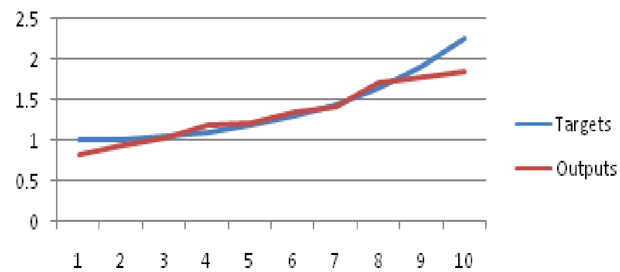

FIGURE 5. Tested outputs inputs with the desired targets(Fractional case)

Training Phase

The NN is trained to estimate the exact solution. The dataset contains 10 exact solutions for training and 10 for testing, solved numerically, using ordinary case(Eq.4) and fractional case(Eq.5).

In the training phase of the NN, the weight matrices between the input and the hidden and output layers are initialized with random values. After repeatedly presenting data of the input samples and desired targets, we then compared the output with the desired outcome, followed by error measurement and weight adjustment. This pattern is repeated until the error rate of the output layer reaches a minimum value. This process is then repeated for the next input value, until all values of the input have been processed. The activation function used is binary-sigmoid. The value of this function ranges between 0 and 1. Whereas, the output layer neuron is estimated using the activation function that features the linear transfer function. The training algorithm used is Gradient descent with momentum back propagation. The exact solutions data are solved manually by using the giving equation and entered as training input data into the NN. The quality of the training sets that enter into the network determines how well the neural network works.

Fig. 2 and 3 show the regressions analysis for both approaches ordinary case (Eq.4) and fractional case (Eq.5) of the trained network. The regressions analysis returns the correlation coefficient R. This coefficient equals to 1 in the output and the target for training; thus, both output and target are very close, which indicates good fit.

Testing Phase

In this phase, the dataset for both cases are prepared in the same manner as in training phase. Depending on training data, 10 exact solutions dataset that were solved numerically, using ordinary case (Eq.4) and fractional case(Eq.5) are used to test the proposed neural network.

Results and Discussions

The experimental results are presented to show the effectiveness of the proposed neural network. The training and testing phases were carried out on a 2.33 GHz Intel (R) Core TM 2Duo CPU 4 GB RAM on Windows 7 platform using MATLAB R2010a. The Fig.4 and Fig.5 show the estimation network results for the testing data for both cases (Ordinary case and Fractional case). Both outputs and targets are very close, these results show that, the neural network works probably and yields error free results.

CONCLUSION

In this paper, we had proposed a new approach for solving the Lane-Emden equations singular initial value problems based on neural network. The results achieved, had been compared with the exact solution. These results show that, the neural network works probably and absolutely no errors were found in the outputs. The parallel processing property of neural network had reduced the computational time which makes this method better than the conventional methods.

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