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EXISTENCE AND UNIQUENESS THEOREM FOR FRACTIONAL DIFFERENTIAL EQUATION WITH INTEGRAL BOUNDARY CONDITION

SHAYMA ADIL MURAD AND SAMIR BASHIR HADID

Abstract. We have investigated the existence and uniqueness solutions of the nonlinear fractional differential equation of an arbitrary order with integral boundary condition. The result is an application of the Schauder fixed point theorem and the Banach contraction principle.

1. Introduction

In recent years, fractional differential equations have been of great interest to many mathematicians. This is due to the development of the theory of fractional calculus itself and the application of such constructions in various fields of science and engineering such as: Control Theory, Physics, Mechanics, Electrochemistry, ... etc. There are many papers discussing the solvability of nonlinear fractional differential equations and the existence of positive solutions of nonlinear fractional differential equations, see the monographs of Kilbas et al. [1], Samko et al [11], and the papers [2, 3, 7, 8, 9, 10] and the references therein. In [5], Benchohra and Ouaar considered the boundary value problem of the fractional differential equation:

$$D^{\alpha}y(t) = f(t, y(t)),\ \ \alpha \in (0, 1]$$
$$y(0) + \mu\int_0^T y(s)ds \ = \ y(T)$$

where D_t^{α} is the Caputo fractional derivative, and $f : [0, 1] \times \Re \to \Re$ is continuous. In [6], Hu and Wang investigated the existence of solution of the nonlinear fractional differential equation with integral boundary condition:

$$D^{\alpha}y(t) = f(t, y(t), D^{\beta}y(t)), \quad t \in (0, 1),$$
$$y(0) = 0, \quad y(1) = \int_0^1 g(s)y(s)ds$$

where $1 < \alpha \leq 2$, $0 < \beta < 1$, and D_t^{α} is the Riemann-Liouville fractional derivative, $f : [0, 1] \times \Re \times \Re \to \Re$, and $g \in L_1[0, 1]$.

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In this paper, we consider the following boundary value problem for the nonlinear fractional differential equation with integral boundary conditions:

$$D^{\alpha}y(t) = f(t, y(t), D^{\beta}y(t)), \quad t \in (0,1), \tag{1}$$

$$y(0) = 0, \quad y(1) = I_0^{1\gamma} y(s) \tag{2}$$

where $1 < \alpha \leq 2$, $0 < \beta < 1$, $0 < \gamma \leq 1$, and D_t^{α} is the Riemann-Liouville fractional derivative $f : [0,1] \times \Re \times \Re \to \Re$. We shall prove that there exists a solution of the boundary value problem (1) and (2), by using the Schauder fixed point theorem and Banach contraction principle. Firstly, we started by driving the corresponding Greens function of (1) and (2). Moreover, we add a certain condition to the above equation in order to obtain a unique solution. The result is more general and contains uniqueness solution. We verify the result by contracting an interesting example.

2. PRELIMINARIES

First of all, we recall some basic definitions

Definition 2.1[11] Let $p, q > 0$, then the beta function $\beta(p,q)$ is defined as:

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx.$$

Remark 2.2[11] For $p, q > 0$, the following identity holds:

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Definition 2.3[11] Let f be a function which is defined almost everywhere (a.e) on $[a,b]$, for $\alpha > 0$, we define:

$${}_a^bD^{-\alpha}f = \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}f(s)ds,$$

provided that the integral (Lebesgue) exists.

Definition 2.4[1] The Riemann-Liouville fractional derivative of order α for a function $f(t)$ is defined by

$${}_a^tD^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1}f(s)ds,$$

where $\alpha > 0$, $n = [\alpha] + 1$ and $[\alpha]$ denote the integer part of α.

Lemma 2.5[4] Let $\alpha > 0$, then

$$D_t^{-\alpha}D_t^{\alpha}y(t) = y(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + ... + c_nt^{\alpha-n},$$

for some $c_i \in \Re$, $i = 0, 1, ..., n-1$, $n = [\alpha] + 1$.

Lemma 2.6 Given $y \in C(0,1)$, $1 < \alpha \leq 2$, $0 < \beta < 1$ and $0 < \gamma \leq 1$. Then, the unique solution of the boundary value problem (1) and (2), is given by:

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s), D^\beta y(s)) ds +$$

$$+ \frac{\zeta t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 \left[\frac{1}{\Gamma(\gamma)} \int_s^1 (1-r)^{\gamma-1} (r-s)^{\alpha-1} dr - (1-s)^{\alpha-1} \right] f(s, y(s), D^\beta y(s)) ds.$$

Proof. By applying Lemma 2.5, equation (1) can be reduced to the equivalent integral equation:

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s), D^\beta y(s)) ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \tag{3}$$

for $c_1, c_2 \in \Re$ and $y(0) = 0$, we can obtain $c_2 = 0$. Then, we can write (3) as

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s), D^\beta y(s)) ds + c_1 t^{\alpha-1},$$

and it follows from $y(1) = I_0^{1\gamma} y(s)$, that

$$c_1 = \frac{\zeta}{\Gamma(\alpha)\Gamma(\gamma)} \int_0^1 \int_0^s (1-s)^{\gamma-1} (s-r)^{\alpha-1} f(r, y(r), D^\beta y(r)) dr ds$$

$$- \frac{\zeta}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, y(s), D^\beta y(s)) ds,$$

$$c_1 = \frac{\zeta}{\Gamma(\alpha)} \int_0^1 \left[\frac{1}{\Gamma(\gamma)} \int_s^1 (1-r)^{\gamma-1} (r-s)^{\alpha-1} dr - (1-s)^{\alpha-1} \right] f(s, y(s), D^\beta y(s)) ds,$$

where $\zeta = \left[1 + \frac{\Gamma(\alpha)}{\Gamma(\alpha+\gamma)} \right]^{-1}$.

Therefore, the solution of (1) and (2) is given by

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s), D^\beta y(s)) ds +$$

$$+ \frac{\zeta t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 \left[\frac{1}{\Gamma(\gamma)} \int_s^1 (1-r)^{\gamma-1} (r-s)^{\alpha-1} dr - (1-s)^{\alpha-1} \right] f(s, y(s), D^\beta y(s)) ds,$$

$$y(t) = \int_0^1 G(t,s) f(s, y(s), D^\beta y(s)) ds. \tag{4}$$

Where G(t,s) is the Green function defined by:

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\zeta t^{\alpha-1}}{\Gamma(\alpha)} \left[\int_s^1 \frac{(1-r)^{\gamma-1}(r-s)^{\alpha-1} dr}{\Gamma(\gamma)} - (1-s)^{\alpha-1} \right], & if \quad 0 \leq s < t \\ \frac{\zeta t^{\alpha-1}}{\Gamma(\alpha)} \left[\int_s^1 \frac{(1-r)^{\gamma-1}(r-s)^{\alpha-1} dr}{\Gamma(\gamma)} - (1-s)^{\alpha-1} \right], & if \quad t \leq s < 1 \end{cases}$$

Thus, we complete the proof.

Next, we define the space $X = \left\{ y(t) \in C[0,1] : D^\beta y(t) \in C[0,1] \right\}$, endowed with the norm :

$$\| y \|_X = \max_{t\in[0,1]} \mid y(t) \mid + \max_{t\in[0,1]} \mid D^{\beta} y(t) \mid$$

is a Banach space.

Lemma 2.7[12] $(X, \| \,.\, \|_X)$ is a Banach space.

Let us set the following notation for convenience:

$$w_1 = \max_{t\in[0,1]} \int_0^1 |G(t,s)a(s)|ds,$$

$$w_2 = \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} a(s)ds + \frac{\zeta}{\Gamma(\gamma)\Gamma(\alpha-\beta)} \int_0^1 \int_s^1 (1-r)^{\gamma-1}(r-s)^{\alpha-1} a(s)drds$$
$$+ \frac{\zeta}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-1} a(s)ds,$$

$$\Lambda_1 = \left(\frac{(1+\zeta)\Gamma(\alpha+\gamma+1) + \zeta\Gamma(\alpha+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\gamma+1)} \right),$$

$$\Lambda_2 = \left(\frac{\alpha\Gamma(\alpha+\gamma+1) + \zeta(\alpha-\beta)\Gamma(\alpha+1) + \zeta(\alpha-\beta)\Gamma(\alpha+\gamma+1)}{\alpha(\alpha-\beta)\Gamma(\alpha+\gamma+1)\Gamma(\alpha-\beta)} \right).$$

3. Main Result

Our main result of this paper is as follows:

Theorem 3.1 Let $f : [0,1] \times \Re \times \Re \to \Re$ be a continuous and assume that the following condition is satisfied:

(H1) There exists a nonnegative function $a(t) \in L_1([0,1])$ such that:
$|f(t,x,y)| \le a(t) + b_1|x|^{\rho_1} + b_2|y|^{\rho_2}$, where $b_1, b_2 \ge 0$ and$0 < \rho_i < 1$ for $i = 1, 2$.

Then the problem(1) and (2) has a solution.

Proof: In order to use Schauder fixed point theorem, to prove our main result, we define $U = \{y(t) \in X : \| y \|_X \le R,\ t \in [0,1]\}$.
As first step, we shall prove that: $T : U \to U$. We have

$$|(Ty)(t)| = \int_0^1 |G(t,s)||f(s,y(s),D^{\beta}y(s))|ds,$$

$$\le \int_0^1 |G(t,s)a(s)|ds + (b_1R^{\rho_1} + b_2R^{\rho_2}) \int_0^1 |G(t,s)|ds,$$

$$\leq \int_0^1 |G(t,s)a(s)|ds + \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \frac{\zeta t^{\alpha-1}}{\Gamma(\alpha)\Gamma(\gamma)} \int_0^1 \int_s^1 (1-r)^{\gamma-1}(r-s)^{\alpha-1} dr ds \right.$$
$$\left. + \frac{\zeta t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds \right) (b_1 R^{\rho_1} + b_2 R^{\rho_2}),$$
$$\leq \int_0^1 |G(t,s)a(s)|ds + \left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\zeta t^{\alpha-1}}{\Gamma(\alpha)\Gamma(\gamma)} \int_0^1 \int_s^1 (1-r)^{\gamma-1}(r-s)^{\alpha-1} dr ds \right.$$
$$\left. + \frac{\zeta t^{\alpha-1}}{\Gamma(\alpha+1)} \right) (b_1 R^{\rho_1} + b_2 R^{\rho_2}).$$

Let $u = \dfrac{r-s}{1-s}$ and $dr = (1-s)du$, we get

$$\leq \int_0^1 |G(t,s)a(s)|ds + \left(\frac{1+\zeta}{\Gamma(\alpha+1)} + \frac{\zeta}{\Gamma(\alpha)\Gamma(\gamma)} \int_0^1 (1-s)^{\alpha+\gamma-1} ds \int_0^1 (1-u)^{\gamma-1} u^{\alpha-1} du \right) (b_1 R^{\rho_1} + b_2 R^{\rho_2}),$$

$$\leq \int_0^1 |G(t,s)a(s)|ds + \left(\frac{1+\zeta}{\Gamma(\alpha+1)} + \frac{\zeta \beta(\gamma,\alpha)}{(\alpha+\gamma)\Gamma(\alpha)\Gamma(\gamma)} \right) (b_1 R^{\rho_1} + b_2 R^{\rho_2}),$$

$$\leq \int_0^1 |G(t,s)a(s)|ds + \left(\frac{1+\zeta}{\Gamma(\alpha+1)} + \frac{\zeta}{\Gamma(\alpha+\gamma+1)} \right) (b_1 R^{\rho_1} + b_2 R^{\rho_2}),$$
$$\leq \int_0^1 |G(t,s)a(s)|ds + \left(\frac{(1+\zeta)\Gamma(\alpha+\gamma+1) + \zeta\Gamma(\alpha+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\gamma+1)} \right) (b_1 R^{\rho_1} + b_2 R^{\rho_2}),$$

$$|(Ty)(t)| \leq w_1 + \Lambda_1 (b_1 R^{\rho_1} + b_2 R^{\rho_2}),$$

and

$$|(D^{\beta}Ty)(t)| \leq \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} |f(s,y(s),D^{\beta}y(s))| ds$$

$$+ \frac{\zeta t^{\alpha-\beta-1}}{\Gamma(\gamma)\Gamma(\alpha-\beta)} \int_0^1 \int_s^1 (1-r)^{\gamma-1}(r-s)^{\alpha-1} |f(s,y(s),D^{\beta}y(s))| dr ds$$

$$+ \frac{\zeta t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-1} |f(s,y(s),D^{\beta}y(s))| ds,$$

$$\leq \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} a(s) ds + \frac{\zeta t^{\alpha-\beta-1}}{\Gamma(\gamma)\Gamma(\alpha-\beta)} \int_0^1 \int_s^1 (1-r)^{\gamma-1}(r-s)^{\alpha-1} a(s) dr ds$$
$$+ \frac{\zeta t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-1} a(s) ds + \left(\int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} ds \right.$$
$$\left. + \frac{\zeta t^{\alpha-\beta-1}}{\Gamma(\gamma)\Gamma(\alpha-\beta)} \int_0^1 \int_s^1 (1-r)^{\gamma-1}(r-s)^{\alpha-1} dr ds + \frac{\zeta t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-1} ds \right) (b_1 R^{\rho_1} + b_2 R^{\rho_2}),$$

$$\leq \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} a(s)ds + \frac{\zeta}{\Gamma(\gamma)\Gamma(\alpha-\beta)} \int_0^1 \int_s^1 (1-r)^{\gamma-1}(r-s)^{\alpha-1} a(s) drds$$

$$+ \frac{\zeta}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-1} a(s)ds + \left(\frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{\zeta t^{\alpha-\beta-1}\beta(\alpha,\gamma)}{(\alpha+\gamma)\Gamma(\gamma)\Gamma(\alpha-\beta)} + \frac{\zeta t^{\alpha-\beta-1}}{\alpha\Gamma(\alpha-\beta)} \right) (b_1 R^{\rho_1} + b_2 R^{\rho_2}),$$

$$\leq \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} a(s)ds + \frac{\zeta}{\Gamma(\gamma)\Gamma(\alpha-\beta)} \int_0^1 \int_s^1 (1-r)^{\gamma-1}(r-s)^{\alpha-1} a(s) drds$$

$$+ \frac{\zeta}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-1} a(s)ds + \left(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{\zeta\Gamma(\alpha)}{\Gamma(\alpha+\gamma+1)\Gamma(\alpha-\beta)} + \frac{\zeta}{\alpha\Gamma(\alpha-\beta)} \right) (b_1 R^{\rho_1} + b_2 R^{\rho_2}),$$

$$|(D^\beta Ty)(t)| \leq \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} a(s)ds + \frac{\zeta}{\Gamma(\gamma)\Gamma(\alpha-\beta)} \int_0^1 \int_s^1 (1-r)^{\gamma-1}(r-s)^{\alpha-1} a(s) drds$$

$$+ \frac{\zeta}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-1}$$

$$+ \left(\frac{\alpha\Gamma(\alpha+\gamma+1) + \zeta(\alpha-\beta)\Gamma(\alpha+1) + \zeta(\alpha-\beta)\Gamma(\alpha+\gamma+1)}{\alpha(\alpha-\beta)\Gamma(\alpha+\gamma+1)\Gamma(\alpha-\beta)} \right) (b_1 R^{\rho_1} + b_2 R^{\rho_2}),$$

Therefore

$$|(D^\beta Ty)(t)| \leq w_2 + \Lambda_2 (b_1 R^{\rho_1} + b_2 R^{\rho_2}),$$

$$||Ty||_X = \max_{t\in[0,1]} |Ty| + \max_{t\in[0,1]} |D^\beta y(t)|,$$

$$||Ty||_X \leq w_3 + \Lambda_3 (b_1 R^{\rho_1} + b_2 R^{\rho_2}) \leq \frac{R}{3} + \frac{R}{3} + \frac{R}{3} \leq R.$$

Where $w_3 = w_1 + w_2$, and $\Lambda_3 = \Lambda_1 + \Lambda_2$. So, we conclude that $||Ty||_X \leq R$. Since $Ty, D^\beta Ty$ are continuous on [0,1], therefor, $T : U \to U$.

Now, we show that T is completely continuous operator. For this purpose we fix $M = \max_{t\in[0,1]} \left| f(s, y(s), D^\beta y(s)) \right|$,

where $y \in U$, and $t, \tau \in [0,1]$, with $t < \tau$, then

$$|T(y)(t) - T(y)(\tau)| = \int_0^1 |G(t,s) - G(\tau,s)| \left| f(s, y(s), D^\beta y(s)) \right| ds,$$

$$\leq M \int_0^1 |G(t,s) - G(\tau,s)| ds,$$

$$\leq M | \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \frac{\zeta t^{\alpha-1}}{\Gamma(\alpha)\Gamma(\gamma)} \int_0^1 \int_s^1 (1-r)^{\gamma-1}(r-s)^{\alpha-1} drds$$

$$- \frac{\zeta t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds - \int_0^\tau \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} ds - \frac{\zeta \tau^{\alpha-1}}{\Gamma(\alpha)\Gamma(\gamma)} \int_0^1 \int_s^1 (1-r)^{\gamma-1}(r-s)^{\alpha-1} drds$$

$$+ \frac{\zeta \tau^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds |,$$

$$\leq \frac{M}{\Gamma(\alpha)} | \int_0^t (t-s)^{\alpha-1} ds - \int_0^\tau (\tau-s)^{\alpha-1} ds | + \frac{M\zeta |t^{\alpha-1} - \tau^{\alpha-1}|}{\Gamma(\alpha)}$$

$$\left(\frac{1}{\Gamma(\gamma)} \int_0^1 \int_s^1 (1-r)^{\gamma-1}(r-s)^{\alpha-1} drds + \int_0^1 (1-s)^{\alpha-1} ds \right),$$

$$\leq \frac{M|t^{\alpha}-\tau^{\alpha}|}{\Gamma(\alpha+1)}+\frac{M\zeta|t^{\alpha-1}-\tau^{\alpha-1}|}{\Gamma(\alpha)}\left(\frac{\beta(\alpha,\gamma)}{(\alpha+\gamma)\Gamma(\gamma)}+\frac{1}{\alpha}\right),$$
$$\leq \frac{M|t^{\alpha}-\tau^{\alpha}|}{\Gamma(\alpha+1)}+M\zeta|t^{\alpha-1}-\tau^{\alpha-1}|\left(\frac{1}{\Gamma(\alpha+\gamma+1)}+\frac{1}{\alpha}\right),$$

and

$$|D^{\beta}T(y)(t)-D^{\beta}T(y)(\tau)|=|\int_0^t\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}f(s,y(s),D^{\beta}y(s))ds$$
$$+\frac{\zeta t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)\Gamma(\gamma)}\int_0^1\int_s^1(1-r)^{\gamma-1}(r-s)^{\alpha-1}f(s,y(s),D^{\beta}y(s))drds$$
$$-\frac{\zeta t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-1}f(s,y(s),D^{\beta}y(s))ds-\int_0^{\tau}\frac{(\tau-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}f(s,y(s),D^{\beta}y(s))ds$$
$$-\frac{\zeta \tau^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)\Gamma(\gamma)}\int_0^1\int_s^1(1-r)^{\gamma-1}(r-s)^{\alpha-1}f(s,y(s),D^{\beta}y(s))drds$$
$$+\frac{\zeta \tau^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-1}f(s,y(s),D^{\beta}y(s))ds|,$$
$$\leq \frac{M}{\Gamma(\alpha-\beta)}|\int_0^t(t-s)^{\alpha-\beta-1}ds-\int_0^{\tau}(\tau-s)^{\alpha-\beta-1}ds|+\frac{M\zeta|t^{\alpha-\beta-1}-\tau^{\alpha-\beta-1}|}{\Gamma(\alpha-\beta)}$$
$$\left(\frac{1}{\Gamma(\gamma)}\int_0^1\int_s^1(1-r)^{\gamma-1}(r-s)^{\alpha-1}drds+\int_0^1(1-s)^{\alpha-1}ds\right),$$
$$\leq \frac{M|t^{\alpha-\beta}-\tau^{\alpha-\beta}|}{\Gamma(\alpha-\beta+1)}+\frac{M\zeta|t^{\alpha-\beta-1}-\tau^{\alpha-\beta-1}|}{\Gamma(\alpha-\beta)}\left(\frac{\beta(\alpha,\gamma)}{(\alpha+\gamma)\Gamma(\gamma)}+\frac{1}{\alpha}\right),$$
$$\leq \frac{M|t^{\alpha-\beta}-\tau^{\alpha-\beta}|}{\Gamma(\alpha-\beta+1)}+\frac{M\zeta|t^{\alpha-\beta-1}-\tau^{\alpha-\beta-1}|}{\Gamma(\alpha-\beta)}\left(\frac{\Gamma(\alpha)}{\Gamma(\alpha+\gamma+1)}+\frac{1}{\alpha}\right).$$

Since the functions $t^{\alpha-\beta-1}$, $t^{\alpha-\beta}$, $t^{\alpha-1}$, and t^{α} are uniformly continuous on $[0,1]$. Therefore, it follows from the above estimate that TU is an equicontinuous set. Also, it is uniformly bounded as $TU\subset U$.
Thus, we conclude that T is completely continuous operator.

Theorem 3.2 Let$f:[0,1]\times\Re\times\Re\rightarrow\Re$ be a continuous. Assume that one of the following conditions is satisfied:

(H2) There exists a constant $k>0$ such that
$|f(t,x,y)-f(t,\bar{x},\bar{y})|\leq k\left(|x-\bar{x}|+|y-\bar{y}|\right)$ for each $t\in[0,1]$ and all $x,y,\bar{x},\bar{y}\in\Re$.

(H3) $\varepsilon=max\left\{k\Lambda_1,k\Lambda_2\right\}$.

Then the problem (1) and (2) has unique solution.

Proof: Let $x,y\in X$, by **(H2)** , we have

$$|T(x)(t)-T(y)(t)|=\int_0^1|G(t,s)||f(s,x(s),D^{\beta}x(s))-f(s,y(s),D^{\beta}y(s))|ds,$$
$$\leq k\int_0^1|G(t,s)|\left[|x(s)-y(s)|+|D^{\beta}x(s)-D^{\beta}y(s)|\right]ds,$$

$$\leq k\left(\frac{1+\zeta}{\Gamma(\alpha+1)}+\frac{\zeta}{\Gamma(\alpha+\gamma+1)}\right)\left[|(x-y)(t)|+|D^{\beta}(x-y)(t)|\right],$$

$$|T(x)(t)-T(y)(t)|\leq k\Lambda_1\left[|(x-y)(t)|+|D^{\beta}(x-y)(t)|\right],$$

and

$$|D^{\beta}T(x)(t)-D^{\beta}T(y)(t)|\leq\int_0^t\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}|f(s,x(s),D^{\beta}x(s))-f(s,y(s),D^{\beta}y(s))|ds$$

$$+\frac{\zeta t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)\Gamma(\gamma)}\int_0^1\int_s^1(1-r)^{\gamma-1}(r-s)^{\alpha-1}|f(s,x(s),D^{\beta}x(s))-f(s,y(s),D^{\beta}y(s))|drds$$

$$+\frac{\zeta t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-1}|f(s,x(s),D^{\beta}x(s))-f(s,y(s),D^{\beta}y(s))|ds,$$

$$\leq k\left[\frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{\zeta t^{\alpha-\beta-1}\beta(\alpha,\gamma)}{(\alpha+\gamma)\Gamma(\alpha-\beta)\Gamma(\gamma)}+\frac{\zeta t^{\alpha-\beta-1}}{\alpha\Gamma(\alpha-\beta)}\right]\left[|(x-y)(t)|+|D^{\beta}(x-y)(t)|\right],$$

$$\leq k\left(\frac{\alpha\Gamma(\alpha+\gamma+1)+\zeta(\alpha-\beta)\Gamma(\alpha+1)+\zeta(\alpha-\beta)\Gamma(\alpha+\gamma+1)}{\alpha(\alpha-\beta)\Gamma(\alpha+\gamma+1)\Gamma(\alpha-\beta)}\right)\left[|(x-y)(t)|+|D^{\beta}(x-y)(t)|\right],$$

$$|D^{\beta}T(x)(t)-D^{\beta}T(y)(t)|\leq k\Lambda_2\left[|(x-y)(t)|+|D^{\beta}(x-y)(t)|\right],$$

then

$$|T(x)(t)-T(y)(t)|\leq k\Lambda_1\left[|(x-y)(t)|+|D^{\beta}(x-y)(t)|\right],$$

$$|D^{\beta}T(x)(t)-D^{\beta}T(y)(t)|\leq k\Lambda_2\left[|(x-y)(t)|+|D^{\beta}(x-y)(t)|\right],$$

$$\|T(x)(t)-T(y)(t)\|_X = \max_{t\in[0,1]}|T(x)(t)-T(y)(t)|+\max_{t\in[0,1]}|D^{\beta}T(x)(t)-D^{\beta}T(y)(t)|,$$

$$\|T(x)(t)-T(y)(t)\|_X\leq\varepsilon\|x-y\|.$$

Hence, we conclude that the problem (1) and (2) has a unique solution by the contraction mapping principle. This ends the proof.
Next we give an example that satisfy the above theorem:

4. Example

Consider the following boundary value problem:

$$\begin{cases} D^{\frac{3}{2}}y(t)=f(t,y(t),D^{\frac{1}{2}}y(t)),\ t\in(0,1), \\ y(0)=0, \qquad y(1)=I_0^{1\,\frac{1}{2}}y(s) \end{cases} \tag{5}$$

Then

$$\zeta=\left[1-\frac{\Gamma(\alpha)}{\Gamma(\alpha+\gamma)}\right]^{-1}=\left[1-\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2}+\frac{1}{2})}\right]^{-1}=8.789425823\,.$$

Let

$$f(t,y(t),D^{\frac{1}{2}}y(t))=\frac{t}{6}+\frac{e^{-t}}{10(1+e^t)}y(t)+D^{\frac{1}{2}}y(t),$$

and $k = \frac{1}{15}$, by a direct calculation, we get

$$\Lambda_1 = \left(\frac{(1+\zeta)\Gamma(\alpha+\gamma+1)+\zeta\Gamma(\alpha+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\gamma+1)} \right) = 11.758835689,$$

and $k\Lambda_1 = 0.78392237928 < 1$.
Similarly

$$\Lambda_2 = \left(\frac{\alpha\Gamma(\alpha+\gamma+1)+\zeta(\alpha-\beta)\Gamma(\alpha+1)+\zeta(\alpha-\beta)\Gamma(\alpha+\gamma+1)}{\alpha(\alpha-\beta)\Gamma(\alpha+\gamma+1)\Gamma(\alpha-\beta)} \right) = 10.754330127139516,$$

and $k\Lambda_2 = 0.7169553418 < 1$.
$\varepsilon = max\{k\Lambda_1, k\Lambda_2\} = max\{0.78392237928, 0.7169553418\} = 0.78392237928 < 1$.
Therefore, by Theorem 3.2, the boundary value problem (5) has a unique solution.

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Shayma Adil Murad
Department of Mathematics, Faculty of Science, Duhok University, Kurdistan, Iraq.

Samir Bashir Hadid
Department of Mathematics and Basic Science, College of Education and Basic Sciences, Ajman University of Science and Technology, UAE.