Solutions of fractional integro-differential equations in L^2 and C-spaces

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Abstract

Sufficient conditions for the existence and uniqueness of L^2 solutions of the fractional integro-differential equation:

$$x^{(\alpha)}(t) = f(t, x(t)) + \int_{t_0}^{t} K(t, s, x(s))ds, \quad \alpha \in \Re, \quad 0 < \alpha \leq 1,$$

with respect to the initial condition $x(t_0) = x_0$ have been obtained. Moreover, we shall prove that under certain conditions on $f(t, x(t))$ and $K(t, s, x(s))$, there exist a unique solution in the space of continuous functions $C[a, b]$. The contraction mapping principle has been used in establishing our main results.

Keywords: Fractional derivatives; integro-differential equations

1 Introduction

Consider the fractional integro-differential equation of the type

$$\begin{aligned} &D_*^\alpha x(t) = f(t, x(t)) + \textstyle\int_{t_0}^{t} K(t, s, x(s))ds, \quad \alpha \in \Re, \quad 0 < \alpha \leq 1, \\ &x(t_0) = x_0, \end{aligned} \tag{1.1}$$

where $\Re$ is the set of real numbers, $I = [t_0, t_0 + a]$, $a > 0$, $f \in C[I \times \Re^n, \Re^n]$, and $K \in C[I \times I \times \Re^n, \Re^n]$, where $\Re^n$ denotes the real $n-$ dimensional Euclidean space, x_0 is a real positive constant, and D_*^α denotes the Caputo fractional derivative operator. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case of $\alpha = 1$, equation (1.1) reduces to the classical integro-differential equation.

Fractional differential equations have gained importance and popularity during the past three decades or so, due to mainly its demonstrated applications in numerous seemingly diverse fields of science and engineering. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with classical integer-order models, in which such effects are in fact neglected. The advantages of fractional derivatives become apparent in modelling mechanical and electrical properties of real materials, as well as in the description of rheological properties of rocks, and in many other fields (see for details [1, 2]).

In recent years, there has been an interest in the study of fractional integro-fractional differential equations. In [3], we used Schauder's fixed-point theorem to obtain local existence, and Tychonov's fixed-point theorem to obtain global existence of solution of (1.1). In [4], we used the successive approximations method and Arzela-Ascoli lemma to obtain existence and uniqueness results of the solution. The existence of extremal (maximal and minimal) solutions of the integro-fractional differential equations (1.1) using comparison principle and Ascoli lemma have been investigated in [5]. While in [6], we obtained an asymptotically stable solutions of (1.1). Finally, in [7] we proved some important results concerning with the corresponding inequalities of the above equation.

On the other hand equation (1.1) is a generalization of the fractional differential equations:

$$\begin{aligned} &D_*^\alpha x(t) = f(t, x(t)), \quad \alpha \in \Re, \quad 0 < \alpha \leq 1, \\ &x(t_0) = x_0, \end{aligned} \tag{1.2}$$

The existence and uniqueness of the solution of (1.2), in additional to some analytical properties and important inequalities, are investigated in [8, 9, 10].

In this paper we impose some conditions on the unique solution of (1.1) that proved by Momani in [3, 4], to be in the space of $L^2[t_0, t_0 + a]$, where $a > 0$ is a real constant. Similarly, we impose some condition on the unique solution of (1.1), to be in the space $C[t_0, t_0 + a]$. Our method in this paper is by using the contraction mapping principle. The results obtained in this paper may be considered as a generalization of the results obtained in [11].

2 Preliminaries and notations

For the concept of fractional derivative we will adopt Caputo's definition which is a modification of the Riemann-Liouville definition and has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variables and their integer order which is the case in most physical processes.

Definition 2.1 *A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p(> \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$, and it is said to be in the space C_μ^m iff $f^{(m)} \in C_\mu$, $m \in N$.*

Definition 2.2 *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as*

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \qquad \alpha > 0, \quad x > 0, \tag{2.1}$$

$$J^0 f(x) = f(x).$$

Properties of the operator J^α can be found in [12-16], we mention only the following:

For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$:

1. $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$,
2. $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$,
3. $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D_*^α proposed by M. Caputo in his work on the theory of viscoelasticity [13].

Definition 2.3 *The fractional derivative of $f(x)$ in the Caputo sense is defined as*

$$D_*^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt, \tag{2.2}$$

for $m-1 < \alpha \leq m, m \in \mathbb{N}, x > 0$, $f \in C_{-1}^m$.

Also, we need here two of its basic properties.

Lemma 2.1 *If $m-1 < \alpha \leq m$, $m \in N$ and $f \in C_\mu^m$, $\mu \geq -1$, then*

$$D_*^\alpha J^\alpha f(x) = f(x),$$

and,

$$J^\alpha D_*^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0.$$

We also need the following result for completeness.

Lemma 2.2 *The initial value problem (1.1) is equivalent to the nonlinear integral equation*

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} f(s, x(s)) ds + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \int_s^t K(\sigma, s, x(s)) d\sigma ds, \tag{2.3}$$

where $0 \leq t_0 < t \leq t_0 + a$. In other words, every solution of the integral equation (2.3) is also a solution of our original initial value problem (1.1), and vice versa.

Proof: It can be proved easily by applying the integral operator (2.1) to both sides of (1.1), and using some classical results from fractional calculus in [17] to get (2.3).

3 The main results

In this section, we shall prove the main theorems, we begin by imposeing some conditions on the unique solution of (1.1) to be in the space of $L^2[t_0, t_0 + a]$, where $a > 0$ is a real constant.

Theorem 3.1 *The integral equation (1.1) has a unique solution $x \in L^2[t_0, t_0 + a]$ provided that:*

(i) f and K satisfy local Lipschitz conditions, with respect to x,

$$|f(t, x_1) - f(t, x_2)| \leq g(s) \|x_1 - x_2\|,$$

and

$$\int_s^t \mid K(\sigma, s, x_1(s)) - K(\sigma, s, x_2(s)) \mid d\sigma \leq h(s) \|x_1 - x_2\|,$$

where the functions g and h are positive and bounded on $0 \leq t_0 < t \leq t_0 + a$, with $g(s) \leq M$, $h(s) \leq N$ and $(t-s)^{\alpha-1}(g(s) + h(s))$ is square-integrable, with:

$$\int_{t_0}^t \int_{t_0}^t |t-s|^{2\alpha-2} (g(s) + h(s))^2 dt ds = P^2 < \infty, \quad (say)$$

(ii) $(t-s)^{\alpha-1}\Big(f(s, 0) + \int_s^t K(\sigma, s, 0) d\sigma\Big)$ is continuous on $[t_0, t_0 + a]$,

(iii) $\Gamma(\alpha) > P$, and $0 < \alpha \leq 1/2$.

Proof: Define an operator T by

$$Tx(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \Big(f(s,x(s))ds + \int_s^t K(\sigma,s,x(s))d\sigma \Big) ds. \tag{3.1}$$

We must first show that T maps $L^2[t_0, t_0+a]$ into itself. Since $L^2[t_0, t_0+a]$ is a Hilbert space, and $x_0 \in L^2[t_0, t_0+a]$, Tx will belong to $L^2[t_0, t_0+a]$ if

$$\int_{t_0}^{t} (t-s)^{\alpha-1} \Big(f(s,x(s))ds + \int_s^t K(\sigma,s,x(s))d\sigma \Big) ds$$

does. Now, using the Lipschitz condition:

$$\begin{aligned}
& |(t-s)^{\alpha-1}||f(s,x(s)) + \int_s^t K(\sigma,s,x(s))d\sigma| - |(t-s)^{\alpha-1}||f(s,0) + \int_s^t K(\sigma,s,0)d\sigma| \\
= \;& |(t-s)^{\alpha-1}||f(s,x(s)) - f(s,0)| + |(t-s)^{\alpha-1}|| \int_s^t K(\sigma,s,x(s))d\sigma - \int_s^t K(\sigma,s,0)d\sigma| \\
\leq \;& |(t-s)^{\alpha-1}|g(s)\|x(s)\| + |(t-s)^{\alpha-1}|h(s)\|x(s)\| \\
\leq \;& |(t-s)^{\alpha-1}|(g(s)+h(s))\|x(s)\|.
\end{aligned}$$

Thus

$$|(t-s)^{\alpha-1}||f(s,x(s)) + \int_s^t K(\sigma,s,x(s))d\sigma| \leq |(t-s)^{\alpha-1}||f(s,0) + \int_s^t K(\sigma,s,0)d\sigma| \\ + |(t-s)^{\alpha-1}|(g(s)+h(s))\|x(s)\|.$$

Therefore,

$$\begin{aligned}
& \int_{t_0}^{t} |(t-s)^{\alpha-1}|| \Big(f(s,x(s)) + \int_s^t K(\sigma,s,x(s))d\sigma \Big) |ds \\
\leq \;& \int_{t_0}^{t} |(t-s)^{\alpha-1}| \Big(f(s,0) + \int_s^t K(\sigma,s,0)d\sigma \Big) |ds + \int_{t_0}^{t} |(t-s)^{\alpha-1}|(g(s)+h(s))||x(s)||ds.
\end{aligned}$$

Each term on the right is square-integrable, the first because it is continuous, the second because it is the result of applying the integral operator with kernal $(t-s)^{\alpha-1}(g(s)+h(s))$ to the function $\| x \| \in L^2[t_0, t_0+a]$ and $(t-s)^{\alpha-1}(g(s)+h(s))$ satisfies $\int_{t_0}^t \int_{t_0}^t |t-s|^{2\alpha-2}(g(s)+h(s))^2 dtds$ is finite, that is because

$$\int_{t_0}^{t} \int_{t_0}^{t} \Big(|t-s|^{2\alpha-2} \Big)^2 (g(s)+h(s))^2 dtds \leq (M+N)^2 \int_{t_0}^{t} \int_{t_0}^{t} |t-s|^{2\alpha-2} dtds < \infty.$$

Thus

$$\int_{t_0}^{t} |(t-s)^{\alpha-1}| \Big(f(s,x(s))ds + \int_s^t K(\sigma,s,x(s))d\sigma \Big) ds,$$

is square-integrable, and we have shows that T maps $L^2[t_0, t_0+a]$ into itself.

Now, we must show that T is a contraction. We have

$$
\begin{aligned}
| Tx_1(t) - Tx_2(t) | &\leq \frac{1}{\Gamma(\alpha)} | \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, x_1(s)) - f(s, x_2(s)) ds | \\
&+ \frac{1}{\Gamma(\alpha)} | \int_{t_0}^{t} (t-s)^{\alpha-1} \int_{s}^{t} (K(\sigma, s, x_1(s)) - K(\sigma, s, x_2(s))) d\sigma ds |, \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} |(t-s)^{\alpha-1}| g(s) \| x_1(s) - x_2(s) \| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} |(t-s)^{\alpha-1}| h(s) \| x_1(s) - x_2(s) \| ds,
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} |(t-s)^{\alpha-1}| (g(s) + h(s)) \| x_1(s) - x_2(s) \| ds, \\
&\leq \frac{1}{\Gamma(\alpha)} \Big(\int_{t_0}^{t} |(t-s)^{2\alpha-2} (g(s) + h(s))^2 ds \int_{t_0}^{t} \| x_1(s) - x_2(s) \|^2 ds \Big)^{1/2}.
\end{aligned}
$$

By the Cauchy-Schwarz inequality. Hence

$$
\begin{aligned}
| Tx_1(t) - Tx_2(t) | &\leq \frac{1}{\Gamma(\alpha)} \Big(\int_{t_0}^{t} |(t-s)^{2\alpha-2} (g(s) + h(s))^2 ds \int_{t_0}^{t} \| x_1(s) - x_2(s) \|^2 ds \Big)^{1/2}, \\
&= \frac{1}{\Gamma(\alpha)} P \| x_1(s) - x_2(s) \| < \| x_1(s) - x_2(s) \|.
\end{aligned}
$$

Thus, T is a contraction on $L^2[t_0, t_0 + a]$, and the equation $Tx = x$, which is just equation (2.3), has exactly one solution, by the fixed point theorem of contraction.

Next, we shall prove that the solution of (1.1) is in the space of continuous functions $C[t_0, t_0 + a]$. Our method is also by using the method of contraction mapping principle and is similar to that obtained in [11] for fractional differential equations.

Theorem 3.2 *The integral equation (2.3) has a unique solution $x \in C[t_0, t_0 + a]$ with the sup norm over $0 \leq t_0 \leq t_0 + a$, provided that*

(i) The function $H(t, s, x(s)) = (t-s)^{\alpha-1} \Big(f(s, x(s)) + \int_s^t | K(\sigma, s, x(s)) d\sigma \Big)$ satisfies a local Lipschitz condition, with respect to the third argument,

$$
|H(t, s, x_1) - H(t, s, x_2)| \leq L(t, s) \| x_1(s) - x_2(s) \|,
$$

where L is a positive function of two variables t and s.

(ii) $\Gamma(\alpha) > sup \Big[\int_{t_0}^{t} L(t, s) ds : \ 0 \leq t_0 \leq r \leq t \leq t_0 + a \Big]$.

Proof: Define an operator T by

$$
Tx(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \Big(f(s, x(s)) + \int_{s}^{t} K(\sigma, s, x(s)) d\sigma \Big) ds. \tag{3.2}
$$

Since f and K are continuous functions, then $\int_{t_0}^{t} (t-s)^{\alpha-1} \Big(f(s, x(s)) + \int_s^t K(\sigma, s, x(s)) d\sigma \Big) ds$ is continuous. This implies that T maps $C[t_0, t_0 + a]$ into itself.

Now, we shall show that T is a contraction. Let x_1 and x_2 be any two functions in $C[t_0, t_0 + a]$, and any $0 \leq t_0 \leq t$,

$$\begin{aligned}
\mid Tx_1(t) - Tx_2(t) \mid &\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \mid f(s, x_1(s)) - f(s, x_2(s)) \mid ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \int_{s}^{t} \mid K(\sigma, s, x_1(s)) - K(\sigma, s, x_2(s)) \mid d\sigma ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \mid H(t, s, x_1(s)) - H(t, s, x_2(s)) \mid ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} L(t, s) \|x_1(s) - x_2(s)\| ds \\
&< \|x_1(s) - x_2(s)\|, \quad (\textit{by assumption (ii)}).
\end{aligned}$$

Thus

$$\mid Tx_1(t) - Tx_2(t) \mid \leq \|x_1(s) - x_2(s)\|,$$

and hence T is a contraction map on $C[t_0, t_0 + a]$, and the equation $Tx = x$, which is just equation (2.3) has exactly one solution, by the fixed point theorem for contraction.

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