International J of Mathematics and Mathematical Sciences, ***No. 45-48, 2503-2507,*** ***2004*** ***, USA.***

Lyapunov Stability Solutions of Integro-Fractional Differential Equations

Samir Hadid [a] , Shaher Momani [b]

[a] *Dept. of Mathematics and Basic Science, Ajman University of Science and Tech., AL-Ain Center, P. O. Box: 17550. UAE, E-mail: sbhadid@yahoo.com*

[b] *Dept. of Mathematics and Comp. Sci., Faculty of Science, P. O. Box: 17551, United Arab Emirates University, Al-Ain, UAE, E-mail: shahermm@yahoo.com*

Abstract

Lyapunov stability and asymptotic stability conditions for the solutions of the integro–fractional differential equations:

$$x^{(\alpha)}(t) = f(t, x(t)) + \int_{t_0}^{t} K(t, s(t), x(s))ds\,, \qquad \alpha \in \Re,\ \ 0 < \alpha \le 1$$

with the initial condition

$$x^{(\alpha-1)}(t_0) = x_0\,,$$

have been investigated. Our methods are applications of Gronwall's Lemma and Schwartz inequality.

MSC: 26A33, 34D20

Keywords: Fractional Derivatives, Lyapunov Stability

1. Introduction

Consider the integro-fractional differential equations of the type:

$$x^{(\alpha)}(t) = f(t, x(t)) + \int_{t_0}^{t} K(t, s(t), x(s))ds\,, \qquad \alpha \in \Re,\ \ 0 < \alpha \le 1 \qquad (1.1)$$

with the initial condition:

$$x^{(\alpha-1)}(t_0) = x_0\,, \qquad (1.2)$$

where $\Re$ is the set of real numbers, $J = [t_0, t_0 + a]$, $f \in C[J \times \Re^n, \Re^n]$, and $K \in C[J \times J \times \Re^n, \Re^n]$, where $\Re^n$ denotes the real n-dimensional Euclidean space, and x_0 is a real constant.

The existence and uniqueness of solution of fractional differential equations, when the integral part in equation (1.1) is identically zero, has been investigated by some authors, see [2], [4], [5], and [6].
In recent papers [8] and [9], we used Schauder's fixed-point theorem to obtain local existence, and Tychonov's fixed-point theorem to obtain global existence of solution of the fractional integro-differential equations (1.1) and (1.2). The existence of extremal (maximal and minimal) solutions of the integro-fractional differential equations (1.1) and (1.2) using comparison principle and Ascoli lemma has been investigated in [10].

In this paper we are concerned with the stability and asymptotic stability, in the sense of Lyapunov, for the solution of the integro-fractional differential equations (1.1) and (1.2). We shall assume that $f(t,0) \equiv 0$ and $K(t,s(t),0) \equiv 0$ for all $t \in J$, so that $x = 0$ is a solution of (1.1).
The zero solution is said to be stable (in the sense of Lyapunov) if, given $\varepsilon > 0$, there exists $\delta > 0$ such that any solution $x(t)$ of (1.1) satisfying $|x(t_0)| < \delta$ for $t = t_0$ also satisfies $|x(t)| < \varepsilon$ for all $t \geq t_0$. The zero solutions is said to be asymptotically stable if, in addition to being stable, $|x(t)| \to 0$ as $t \to \infty$.
Our result is a generalization of Hadid [3], in which it was shown that under certain conditions on f the zero solution of the IVP:

$$x^{(\alpha)}(t) = f(t, x(t)), \qquad \alpha \in \Re, \ 0 < \alpha \leq 1 \tag{1.3}$$

with the initial condition

$$x^{(\alpha-1)}(t_0) = x_0, \tag{1.4}$$

is stable and hence it is a asymptotically stable.

Next we set forth definitions and lemmas to be used in this paper. For proofs and details see Barrett [1], Bassam [2], and Hadid [4].

Definition 1.1. Let f be a function which is defined almost everywhere (a.e.) on $[a,b]$. For $\alpha > 0$, we define

$$\overset{b}{\underset{a}{I}}{}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} f(s)ds, \tag{1.5}$$

provided that this integral (Lebesgue) exists, where Γ is the Gamma function.

Lemma 1.1. The initial value problem (1.1) and (1.2) is equivalent to the nonlinear integral equation

$$x(t) = \frac{x_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, x(s))ds +$$

$$\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\int_{s}^{t}K(\sigma,s,x(s))d\sigma\,ds \qquad (1.6)$$

where $0<t_0<t\le t_0+a$. In other words, every solution of the integral (1.6) is also a solution of our original initial value problem (1.1) and (1.2), and vice versa.

Proof. It can be proved easily by applying the integral operator (1.5) to both sides of (1.1), as we did in [3] , and using some classical results from fractional calculus in [1] to get (1.6).

Lemma 1.2 (Gronwall's Lemma). Let $u(t)$ and $v(t)$ be non-negative continuous function on some interval $t_0 \le t \le t_0 + a$. Also, let the function $f(t)$ be positive, continuous, and monotonically non-decreasing on $t_0 \le t \le t_0 + a$ and satisfy the inequality

$$u(t)\le f(t)+\int_{t_0}^{t}u(s)v(s)ds, \qquad (1.7)$$

then, we have

$$u(t)\le f(t)\exp\left(\int_{t_0}^{t}v(s)ds\right) \quad \text{for} \quad t_0\le t\le t_0+a.$$

Proof. See Plaat [11].

2. Stability Conditions

In this section, we shall prove our main results, we shall discuss the stability and asymptotic stability of the solution of (1.1) satisfying (1.2).

Theorem 2.1. Let the function f satisfies the inequality

$$|f(t,x(t))|\le\gamma(t)|x(t)|, \qquad (2.1)$$

and K satisfies

$$\left|\int_{s}^{t}K(\sigma,s,x(s))d\sigma\right|\le\delta(t)|x(t)|, \qquad (2.2)$$

where $\gamma(t)$ and $\delta(t)$ are continuous non-negative functions such that:

$$\sup\int_{t_0}^{t}(t-s)^{\alpha-1}[\gamma(s)+\delta(s)]ds<\infty. \qquad (2.3)$$

Then every solution $x(t)$ of (1.1) satisfies

$$|x(t)| \le \frac{|x_0|}{\Gamma(\alpha)}(t-t_0)^{\alpha-1} \exp\left\{ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1}[\gamma(s)+\delta(s)]ds < \infty \right\} \tag{2.4}$$

Proof. For $0 \le t_0 < s < t \le t_0 + a$, it follows from (1.6) that

$$\Gamma(\alpha)|x(t)| \le |x_0|(t-t_0)^{\alpha-1} + \int_{t_0}^{t}(t-s)^{\alpha-1}\gamma(s)|x(s)|ds + \int_{t_0}^{t}(t-s)^{\alpha-1}\delta(s)|x(s)|ds$$

By combined the integrals on the R.H.S., we get

$$\Gamma(\alpha)|x(t)| \le |x_0|(t-t_0)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}[\gamma(s)+\delta(s)]\Gamma(\alpha)|x(s)|ds$$

By Gronwell's lemma, we obtain:

$$\Gamma(\alpha)|x(t)| \le |x_0|(t-t_0)^{\alpha-1} \exp\left\{ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t}(t-s)^{\alpha-1}[\gamma(s)+\delta(s)]ds \right\}$$

Therefore

$$|x(t)| \le \frac{|x_0|(t-t_0)^{\alpha-1}}{\Gamma(\alpha)} \exp\left\{ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t}(t-s)^{\alpha-1}[\gamma(s)+\delta(s)]ds \right\}$$

Hence the theorem is proved.

Corollary 2.1. If

$$\int_{t_0}^{t}(t-s)^{\alpha-1}[\gamma(s)+\delta(s)]ds = O\left((t-t_0)^{\alpha-1}\right)$$

then

$$|x(t)| \le C_0\left((t-t_0)^{\alpha-1}\right), \tag{2.5}$$

where C_0 is a positive constant, and hence the solution of (1.1) and (1.2) is asymptotically stable

Corollary 2.2. We can easily show, from (2.5), that

$$x(t) \in L^2(t_0,\infty) \quad \text{for all} \quad 0 < \alpha < \frac{1}{2}$$

Next, we shall prove another important stability result. The result is in connection with α, the method we shall use is an application of Schwartz inequality.

Theorem 2.2. Assume that:

(i) The function f be in $L^2(t_0,\infty)$ as a function of t,

$(ii)\ K\big(t, s(t), x(s)\big) \ = O\left((t-s)^{\alpha - \frac{3}{2}} \right)$ (2.6)

Then, for $0 < \alpha < \frac{1}{2}$, the zero solution of (1.1) and (1.2) is asymptotically stable.

Proof. For $0 \le t_0 < s < t \le t_0 + a$, it follows from (1.6) that:

$$\Gamma(\alpha)\, x(t) = x_0 (t-t_0)^{\alpha-1} + \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, x(s)) ds + \\ + \int_{t_0}^{t} (t-s)^{\alpha-1} \int_{s}^{t} K(\sigma, s, x(s)) d\sigma ds .$$

By applying the absolute value, we get:

$$\Gamma(\alpha)\ |x(t)| \le\ |x_0|\,(t-t_0)^{\alpha-1} + \int_{t_0}^{t} (t-s)^{\alpha-1}\, |f(s, x(s))| ds \\ + \int_{t_0}^{t} (t-s)^{\alpha-1} \int_{s}^{t} |K(\sigma, s, x(s))| d\sigma ds .$$

By Schwartz inequality, we obtain:

$$\Gamma(\alpha)\ |x(t)| \le\ |x_0|\,(t-t_0)^{\alpha-1} + \left(\int_{t_0}^{t} (t-s)^{2\alpha-2} ds \right)^{\frac{1}{2}} \left(\int_{t_0}^{t} |f(s, x(s))|^2 ds \right)^{\frac{1}{2}} \\ + \left(\int_{t_0}^{t} (t-s)^{2\alpha-2} ds \right)^{\frac{1}{2}} \left(\int_{t_0}^{t} \left(\int_{s}^{t} |K(\sigma, s, x(s))| d\sigma \right)^2 ds \right)^{\frac{1}{2}}$$

Now, by using (i) and (ii) in the statement of the theorem and integrate, we obtain:

$$\Gamma(\alpha)\ |x(t)| \le\ |x_0|\,(t-t_0)^{\alpha-1} + C_1 (t-t_0)^{\alpha-\frac{1}{2}} + C_2 (t-t_0)^{\alpha-\frac{1}{2}} \ , \qquad (2.7)$$

where C_1 and C_2 are positive constants (we can calculate them easy).
By (2.7), and for $0 \le t_0 < s < t \le t_0 + a$, we have:

$$|x(t)| \le\ (t-t_0)^{\alpha-1} \left[\frac{|x_0|}{\Gamma(\alpha)} (t-t_0)^{-\frac{1}{2}} + C_1 + C_2 \right]. \qquad (2.8)$$

Therefore, from (2.8) we obtain that the zero solution of (1.1) satisfying (1.2) is asymptotically stable. Hence the theorem is proved.

References

[1] J. Barrett, Differential equations of non-integer order, *Canad. J. Math.,* **6** (1954), 529-541.

[2] M. Bassam, Some existence theorems on D. E. of generalized order, *J. fur die reine und angewandte Mathematik*, Band **218** (1965), 70-78.

[3] S.B.Hadid and J.AL-Shamani, Liapunov stability of D. E. of non-integer order, *Arab J. Math.,* Vol. 5, No. 1 and 2 (1986), 5-17.

[4] S. B. Hadid, Local and global existence theorems on differential equations of non-integer order, *J. Fractional Calculus*, Vol. 7, May (1995), 101-105.

[5] S. B. Hadid, B. Masaedeh, and S. Momani, On the existence of maximal and minimal solutions of D. E. of non-integer order, *J. Fractional Calculus*, Vol.9, May (1996), 41-44.

[6] S. B. Hadid, A. A. Ta'ani, and S. Momani, Some existence theorems on D. E. of generalized order through a fixed-point theorem, *J. Fractional Calculus,* Vol. 9, May (1996), 45-49.

[7] V. Lakshmikantham and M. R. Rao, Theory of integro-differential equations, *Gordon Breach Sci. Pub*., Lausanne, 1995.

[8] S. Momani, Local and global existence theorems on fractional ntegro-differential equations, *J. Fractional Calculus,* Vol. 18, November (2000), 81-86.

[9] S. M. Momani, Some existence theorems on fractional integro-differential equations, accepted for publication in *Abhath Al-Yarmouk Journal.*

[10] S. M. Momani and R. El-Khazali, On the existence of extremal solutions of the fractional integro-differential equations, *J. Fractional Calculus*, Vol.18, November (2000), 87-92.

[11] O. Plaat, Ordinary Differential Equations, *Holden-Day,* Inc. San Francisco, 1971.