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On the Continuous Dependence of Solutions of Integro-Fractional Differential Equations with Respect to Initial Conditions

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Abstract-The continuous dependence of the integro–fractional differential equations:

$$x^{(\alpha)}(t) = f(t, x(t)) + \int_{t_0}^{t} K(t, s(t), x(s))ds, \qquad \alpha \in \Re, \quad 0 < \alpha \le 1$$

with respect to initial conditions have been investigated. We obtain two results concerning the existence and continuity of solutions with respect to initial conditions. Moreover, we obtain a result that involves estimating a function satisfying an integro–fractional differential inequality by the maximal solution of the corresponding equation.

Keywords-Fractional derivatives, Integro-differential equations.

1. INTRODUCTION

Consider the integro-differential equations with non-integer order of the type:

$$x^{(\alpha)}(t) = f(t, x(t)) + \int_{t_0}^{t} K(t, s(t), x(s))ds, \qquad \alpha \in \Re, \quad 0 < \alpha \le 1 \tag{1.1}$$

with the initial condition:

$$x^{(\alpha-1)}(t_0) = x_0, \tag{1.2}$$

where $\Re$ is the set of real numbers, $J = [t_0, t_0 + a]$, $a > 0$, $f \in C[J \times \Re^n, \Re^n]$, and $K \in C[J \times J \times \Re^n, \Re^n]$, where $\Re^n$ denotes the real n-dimensional Euclidean space, and x_0 is a real positive constant.

In recent years, there has been an interest in the study of integro-fractional differential equations. In [8], we used Schauder's fixed-point theorem to obtain local existence, and Tychonov's fixed-point theorem to obtain global existence of solution

of (1.1) and (1.2). In [9], we used the successive approximations method and Arzela-Ascoli lemma to obtain existence and uniqueness of solution of equation (1.1) and (1.2). The existence of extremal (maximal and minimal) solutions of the integro-fractional differential equations (1.1) and (1.2) using comparison principle and Ascoli lemma have been investigated in [10]. While in [6], we obtained an asymptotically stable solutions of (1.1) and (1.2). Finally, in [11] we proved some important results concerning inequalities and (1.1) and (1.2).
On the other hands equations (1.1) and (1.2) are a generalization of the fractional differential equations:

$$x^{(\alpha)}(t) = f(t, x(t)), \qquad \alpha \in \Re, \;\; 0 < \alpha \le 1, \tag{1.3}$$

with the initial condition:

$$x^{(\alpha-1)}(t_0) = x_0. \tag{1.4}$$

The existence and uniqueness of the solution of (1.3) and (1.4), in additional to some analytical properties and important inequalities, are investigated in [4] and [5].

In this paper, we shall consider the problem of continuity of solutions of the integro-fractional differential equations (1.1) with respect to initial values (t_0, x_0).
Also, we obtain a theorem involving estimating a function satisfying an integro–fractional differential inequality by the maximal solution of the corresponding equation. The prove of this theorem is different from that in [10], in which we obtain a similar theorem.
The results obtained in this paper may be considered as a generalization of the integro-differential equations in [7].

We shall adopt the definitions and notations used in [1] and [2].

Definition (1.1): Let f be a function which is defined almost everywhere (a.e.) on $[a,b]$. For $\alpha > 0$, we define

$$\overset{b}{\underset{a}{I}}{}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} f(s)\,ds, \tag{1.5}$$

provided that this integral (Lebesgue) exists, where Γ is the Gamma function.

We also need the following two results for completeness.

Lemma (1.1): The initial value problem (1.1) and (1.2) is equivalent to the nonlinear integral equation

$$x(t)=\frac{x_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s,x(s))ds+$$
$$\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\int_{s}^{t}K(\sigma,s,x(s))d\sigma\, ds\,, \qquad (1.6)$$

where $0\le t_0<t\le t_0+a$.

In other words, every solution of the integral (1.6) is also a solution of our original initial value problem (1.1) and (1.2), and vice versa.

Proof: It can be proved easily by applying the integral operator (1.5) to both sides of (1.1), as we did in [4] , and using some classical results from fractional calculus in [1] to get (1.6).

Theorem (1.1): Assume that

(i) $f\in C\left[J\times\Re^n,\Re^n\right]$, $K\in C\left[J\times J\times\Re^n,\Re^n\right]$ and $\int_s^t\left|K(\sigma,s,x(s))\right|\,d\sigma\le N$ for $0\le t_0<s<t\le t_0+a$, $x\in\Omega=\{\Phi\in C[J,\Re^n]:\ \Phi^{(\alpha-1)}(t_0)=x_0$, and $\left|\Phi(t)-\frac{x_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1}\right|\le b\ \}$;

(ii) $\left|f(t,x)-f(t,y)\right|\le g(t,|x-y|)$ and $\left|K(t,s,x)-K(t,s,y)\right|\le H(t,s,|x-y|)$, where $g\in C[J\times[0,2b],\Re_+]$, $H\in C[J\times J\times[0,2b],\Re_+]$, $g(t,0)\equiv 0$, $H(t,s,0)\equiv 0$, $g(t,u)$ and $H(t,s,u)$ are non-decreasing in u for each $(t,s)\in J\times J$ and $\int_s^t\left|H(\sigma,s,u(s))\right|\,d\sigma\le N_0$ for $0\le t_0<s<t\le t_0+a$, $u\in\Omega^0=\{u\in C[J,\Re_+]:\ |u(t)|\le 2b\ \}$;

(iii) the comparison equation

$$u^{(\alpha)}(t)=g(t,u(t))+\int_{t_0}^{t}H(t,s,u(s))ds,\quad \alpha\in\Re,\quad 0<\alpha\le 1,$$
$$u^{(\alpha-1)}(t_0)=u_0\ge 0\,.$$

admits only the trivial solution.

Then the successive approximations defined by

$$x_{n+1}(t)=\frac{x_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s,x_n(s))ds+$$
$$\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\int_{s}^{t}K(\sigma,s,x_n(s))d\sigma\, ds\,,$$

exists on $0 \le t_0 < t \le t_0 + \beta$, where $\beta^{\alpha} = \min\left\{a, \frac{b\alpha\Gamma(\alpha)}{M+N}\right\}$, as continuous functions and converge uniformly on this interval to the solution $x(t)$ of (1.1) and (1.2).

Proof: The proof of the above theorem is found in [9].

2. THE MAIN THEOREMS

The this section, we shall prove our main results, we begin by proving a theorem concerned with estimating a function satisfying an integro–fractional differential inequality by the maximal solution of the corresponding integro–fractional differential equation.

Theorem (2.1): Assume that

$g \in C[\Re_+^2, \Re]$, $H \in C[\Re_+^3, \Re]$, $H(t,s,u)$ is non-decreasing function in u for each (t,s) and for $t > t_0$,

$$m^{(\alpha)}(t) \le g(t, m(t)) + \int_{t_0}^{t} H(t,s,m(s))ds, \tag{2.1}$$

where $m \in C[\Re_+, \Re]$. Sup pose that $\gamma(t)$ is the maximal solution of

$$\left.\begin{aligned} &u^{(\alpha)}(t) = g(t,u(t)) + \int_{t_0}^{t} H(t,s,u(s))ds, \quad \alpha \in \Re, \quad 0 < \alpha \le 1, \\ &u^{(\alpha-1)}(t_0) = u_0 \ge 0, \end{aligned}\right\} \tag{2.2}$$

existing on $0 \le t_0 < t \le t_0 + a$. Then

$$m(t) \le \gamma(t), \quad t \ge t_0, \tag{2.3}$$

provided that

$$m^{(\alpha-1)}(t_0) \le u^{(\alpha-1)}(t_0) = u_0. \tag{2.4}$$

Proof: Let $t_0 > \tau$. By Theorem (1) in [10], the maximal solution $\gamma(t,\varepsilon)$ of

$$\left.\begin{aligned} &u^{(\alpha)}(t) = g(t,u(t)) + \varepsilon + \int_{t_0}^{t} H(t,s,u(s))ds, \\ &u^{(\alpha-1)}(t_0) = u_0 + \varepsilon, \end{aligned}\right\} \tag{2.5}$$

exists on $[t_0, \tau]$ for all $\varepsilon > 0$ sufficiently small, and

$$\gamma(t) = \lim_{\varepsilon \to 0} \gamma(t,\varepsilon) \text{ uniformly on } [t_0, \tau]. \tag{2.6}$$

Using (2.5) and (2.1) and applying the inequality (2.4), we find that

$$m(t) < \gamma(t,\varepsilon), \quad t \in [t_0, \tau].$$

The last inequality together with (2.6) proves the assertion (2.3) of the theorem.

To consider the problem of continuity of solutions of (1.1) with respect to initial values (t_0, x_0), we need the following result

Theorem (2.2): Assume that

(*i*) $f \in C[J \times \Re^n, \Re^n]$ and let $G(t,\gamma) = \max\limits_{\left|x - \frac{x_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1}\right| \le \gamma} |f(t,x)|$;

(*ii*) $K \in C[J \times J \times \Re^n, \Re^n]$ and let $H(t,s,\gamma) = \max\limits_{\left|x - \frac{x_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1}\right| \le \gamma} |K(t,s,x)|$;

(*iii*) $\gamma^*(t,t_0,0)$ is the maximal solution of

$$u^{(\alpha)}(t) = G(t,u(t)) + \int_{t_0}^{t} H(t,s,u(s))ds,$$

$$u^{(1-\alpha)}(t_0) = 0.$$

If $x(t) = x(t,t_0,x_0)$ is any solution of (1.1) and (1.2) then

$$\left| x(t,t_0,x_0) - \frac{x_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1} \right| < \gamma^*(t,t_0,0), \qquad t \ge t_0 .$$

Proof:

Define the function

$$v(t) = \left| x(t) - \frac{x_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1} \right| . \tag{2.7}$$

Then from (2.7) we have

$$v(t) = \left| x(t) - \frac{x_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1} \right| \le |x(t)| , \tag{2.8}$$

and therefore

$$v^{(\alpha)}(t) \le \left| x^{(\alpha)}(t) \right| = \left| f(t,x(t)) + \int_{t_0}^{t} K(t,s,x(s))ds \right|$$

$$\le \max_{\left|x - \frac{x_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1}\right| \le v(t)} |f(t,x)| + \int_{t_0}^{t} \max_{\left|x - \frac{x_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1}\right| \le v(s)} |K(t,s,x(s))| ds$$

$$= G(t,v(t)) + \int_{t_0}^{t} H(t,s(t),v(s))ds$$

This implies by Theorem(2.1) that

$$v(t) = \left| x(t,t_0,x_0) - \frac{x_0}{\Gamma(\alpha)}(t-t_0)^{\alpha-1} \right| \le \gamma^*(t,t_0,0), \qquad t \ge t_0 .$$

And hence the theorem is proved.

Now we are going to prove the continuous dependence of solutions $x(t,t_0,x_0)$ of (1.1) with respect to initial values (t_0,x_0) .

Theorem (2.3): Let the hypothesis of Theorem (1.1) be satisfied. Suppose that the solutions $u(t,t_0,u_0)$ of (2.2) through every point (t_0,u_0) are continuous with respect to initial conditions (t_0,u_0) . Then the solutions $x(t,t_0,x_0)$ of (1.1) are unique and continuous with respect to the initial values (t_0,x_0) **.**

Proof:

Since the uniqueness follows from Theorem (1.1), we have to prove the continuity part only. To that end, let $x(t,t_0,x_0)$ and $y(t,t_0,x_0)$ be the solutions of (1.1) through (t_0,x_0) and (t_0,y_0) respectively. Define $v(t) = |x(t,t_0,x_0) - y(t,t_0,y_0)|$; then condition (ii) in Theorem (2.2) implies the integro-fractional differential inequality

$$v^{(\alpha)}(t) \le g(t,v(t)) + \int_{t_0}^{t} H(t,s(t),v(s))ds .$$

By Theorem (2.1), we obtain $v(t) \le \gamma(t,t_0,\frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)}|x_0 - y_0|), t \ge t_0$, where $\gamma(t,t_0,\frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)}|x_0 - y_0|)$ is the maximal solution of (2.2), such that $u^{(\alpha-1)}(t_0) = |x_0 - y_0|$. Since the solution $u(t,t_0,u_0)$ of (2.2) is assumed to be continuous with respect to the initial values, it follows that $\underset{x_0 \to y_0}{Lim}\, \gamma(t,t_0,\frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)}|x_0 - y_0|) = \gamma(t,t_0,0)$. By hypothesis $\gamma(t,t_0,0) \equiv 0$. This in view of the fact the definition of $v(t)$ yields $\underset{x_0 \to y_0}{Lim}\, x(t,t_0,x_0) = y(t,t_0,x_0)$, which shows the continuity of $x(t,t_0,x_0)$ with respect to x_0 **.**

We shall next prove the continuity with respect to t_0 .

If $x(t,t_0,x_0)$ and $y(t,t_1,x_0)$, $t_1 > t_0$, are the solutions of (1.1) through (t_0,x_0) and (t_1,x_0) respectively then, as before, we obtain the inequality

$$v^{(\alpha)}(t) \le g(t, v(t)) + \int_{t_0}^{t} H(t, s(t), v(s))ds ,$$

where $v(t) = |x(t, t_0, x_0) - y(t, t_1, x_0)|$. Also $v(t_1) = |x(t_1, t_0, x_0) - x_0|$. Hence, by Theorem(2.2), $v(t_1) \le \gamma^*(t_1, t_0, 0)$, and consequently $v(t) \le \tilde{\gamma}(t)$, $t_1 > t_0$, where $\tilde{\gamma}(t) = \tilde{\gamma}(t, t_1, \gamma*(t_1, t_0, 0))$ is the maximal of (2.2) through $(t_1, \gamma*(t_1, t_0, 0))$. Since $\gamma*(t_0, t_0, 0) \equiv 0$, we have

$$\lim_{t_1 \to t_0} \tilde{\gamma}(t, t_1, \gamma*(t_1, t_0, 0)) = \tilde{\gamma}(t, t_0, 0),$$

and by hypothesis $\tilde{\gamma}(t, t_0, 0)$ is identically zero, thus proving the continuity of $x(t, t_0, x_0)$ with respect to t_0. The proof of the theorem is complete.

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REFERENCES

1. J. Barrett, Differential equations of non-integer order, *Canad. J. Math*., **6** 529-541, (1954).
2. M. Bassam, Some existence theorems on differential equations of generalized order, *J. fur die reine und angewandte Mathematik*, Band **218,** 70-78, (1965).
3. N. Dunford and J. Schwartz, *Linear operators, part 1*, Interscince, New-Yourk, 1958.
4. S. B. Hadid and J. AL-Shamani, Liapunov stability of differential equations of non-integer order, *Arab J. Math.*, Vol. **5**, No. 1 and 2, 5-17, (1986).
5 S. B. Hadid, Local and global existence theorems on differential equations of non-integer order, *J. Fractional Calculus*, Vol. **7,** May, 111-115, (1995).

6. S. B. Hadid and S. M. Momani, Liapunov stability solutions of integro-fractional differential equations, Submitted for *J. Comput. Appl. Math*.
7. V. Lakshmikantham and M. R. Rao, *Theory of integro-differential equations*, Gordon Breach Sci. Pub., Lausanne, 1995.
8. S. Momani; Local and global existence theorems on fractional Integro-differential equations, *J. Fractional Calculus*, Vol. **18**, November, 81-86, (2000).
9. S. M. Momani; Some existence fractional integro-differential equations, accepted for publication in *Abhath Al-Yarmouk Journal*, Yarmouk University.
10. S. M. Momani and R. El-Khazali, On the existence of extremal solutions of the fractional integro-differential equations, *J. Fractional Calculus*, Vol. **18**, November, 87-92, (2000).
11. S. M. Momani and S. B. Hadid, On the inequalities of integro-fractional differential equations, Submitted for *Arch. Inequal. Appl. J.*
12. O. Plaat, *Ordinary Differential Equations*, Holden-Day, Inc. San Francisco, 1971.
13. M. R. Rao, *Ordinary Differential Equations*, East-West Press PVT limited, 1980.